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Exactly Solvable $su(N)$ Mixed Spin Ladders[1]

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It is shown that solvable mixed spin ladder models can be constructed from $su(N)$ permutators. Heisenberg rung interactions appear as chemical potential terms in the Bethe Ansatz solution. Explicit examples given are a mixed spin-$\frac{1}{2}$ spin-1 ladder, a mixed spin-$\frac{1}{2}$ spin-$\frac{3}{2}$ ladder and a spin-1 ladder with biquadratic interactions.

KEY WORDS: Exactly solved models; Spin chains; Spin ladders; Bethe Ansatz.

1. INTRODUCTION

It is well known that exact solutions of realistic models in statistical mechanics are of immense importance. Beyond physical insights, they provide benchmarks against which approximate techniques may be tested, and in some cases, a stimulus to further research through their strong predictive power. Let us recall a quote from Professor Baxter's book:[(1)]

> Basically, I suppose the justification for studying these lattice models is very simple: they are relevant and they can be solved, so why not do so and see what they tell us?

This is precisely the spirit of our recent work on ladder models, which are systems of coupled quantum spin chains. A number of exactly solved spin ladders have been found.[3] Here by exactly solved we mean integrable

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[3] References 2–12 provide only a partial list.

in the Yang–Baxter sense, with a corresponding Bethe Ansatz solution.[4] A particularly neat construction is that given in refs. 5, 7, and 11. There it is shown that integrable spin models constructed from the fundamental representation of the algebras $su(N)$, $so(N)$ and $sp(N)$, where $N=2^n$, can be reinterpreted as n-leg spin-$\frac{1}{2}$ ladder models. Here we show that mixed spin integrable ladder models can be constructed from the $su(N)$ family for any N.

The 2-leg spin-$\frac{1}{2}$ ladder model of Wang[(5)] is of considerable interest. It differs from the experimentally significant[(15, 16)] spin-$\frac{1}{2}$ Heisenberg ladder through a four-body spin interaction, which is necessary to make the model solvable. Such a four-spin interaction term has been introduced on physical grounds[(17)] (see also ref. 18). In Wang's model, the effect of this term is to shift the critical value of the rung coupling J at which the model becomes massive. In the integrable model the Heisenberg rung coupling breaks the underlying $su(4)$ symmetry and appears as a chemical potential term in the Bethe Ansatz solution. Wang's model was shown to be part of an $su(N)$ family of ladder models.[(7)] The phase diagram has been calculated for the 3-leg ladder model, which includes the 3-leg spin tube.[(19, 20)] These calculations reveal magnetisation plateaus[(21, 22)] in the presence of a magnetic field.[(20)] Moreover, the exact magnetic phase diagrams are seen to be in qualitative agreement with those of the n-leg Heisenberg ladders.[(22)]

The solvable ladder models are thus seen to be of relevance. This is largely because they incorporate the same Heisenberg rung interactions as the Heisenberg ladders. It is well known that the rung interactions drive the physics of the ladder systems.[(15, 16)] Here we present new families of solvable mixed spin ladders.

This paper is arranged as follows. In Section 2 we review the basic ingredients of the $su(N)$ lattice models and their Bethe Ansatz solution. Then in Section 3 we construct the related mixed spin ladder models. In Section 4 we consider the rung interactions which preserve integrability. Some explicit examples are given in Section 5.

2. *su(N)* MODELS

We recall that an integrable spin-S chain can be constructed from a solution of the Yang–Baxter equation. Here we briefly review this construction for the case of the $su(N)$ algebras. The Chevalley generators in the fundamental representation of the $su(N)$ algebra are given by

$$X_\alpha^+ = E_{\alpha,\alpha+1}, \qquad X_\alpha^- = E_{\alpha+1,\alpha}, \qquad H_\alpha = E_{\alpha\alpha} - E_{\alpha+1,\alpha+1} \tag{1}$$

[4] Another class of ladder model can be constructed with matrix product groundstates, see for example, refs. 13, 14 and references therein.

for $1 \leqslant \alpha \leqslant N-1$. The $N \times N$ matrices $E_{\alpha\beta}$ have a 1 in the αth row and βth column and zeros everywhere else. These generators satisfy the defining relations of $su(N)$,

$$[X_\alpha^+, X_\beta^-] = \delta_{\alpha\beta} H_\alpha, \qquad [H_\alpha, X_\beta^\pm] = \pm a_{\alpha\beta} X_\beta^\pm, \qquad [H_\alpha, H_\beta] = 0 \tag{2}$$

Here, $a_{\alpha\beta}$ are the Cartan matrix elements corresponding to the A_{N-1} Dynkin diagram, given by

$$a_{\alpha\beta} = \begin{cases} 2 & \alpha = \beta \\ -1 & \alpha = \beta \pm 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

From the Chevalley generators one may construct a spin-$(N-1)/2$ operator given by

$$(S^\pm)^{(N)} = \sum_{\alpha=1}^{N-1} \sqrt{\alpha(N-\alpha)}\, X_\alpha^\pm, \qquad (S^z)^{(N)} = \tfrac{1}{2} \sum_{\alpha=1}^{N-1} \alpha(N-\alpha)\, H_\alpha \tag{4}$$

where $S^\pm = S^x \pm \mathrm{i} S^y$. These satisfy the $su(2)$ relations.

In terms of the $su(N)$ elements a solution of the Yang–Baxter equation is given by

$$P^{(N)} = \sum_{\alpha, \beta=1}^{N} E_{\alpha\beta} \otimes E_{\beta\alpha} \tag{5}$$

It follows that the following Hamiltonian is integrable,

$$H = \sum_{i=1}^{L} P_{i,i+1}^{(N)} \tag{6}$$

where $P_{i,j}^{(N)}$ acts as the permutator (5) on the ith and jth factor in the Hilbert space $\bigotimes_{i=1}^{L} \mathbb{C}_i^N$ and as the identity everywhere else. H can be diagonalized using the Bethe Ansatz. The Bethe Ansatz equations are well known,[23] and given by

$$\begin{aligned} \left(\frac{\lambda_j^{(1)} - \mathrm{i}/2}{\lambda_j^{(1)} + \mathrm{i}/2}\right)^L &= \prod_{k \neq j}^{M_1} \frac{\lambda_j^{(1)} - \lambda_k^{(1)} - \mathrm{i}}{\lambda_j^{(1)} - \lambda_k^{(1)} + \mathrm{i}} \prod_{k=1}^{M_2} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + \mathrm{i}/2}{\lambda_j^{(1)} - \lambda_k^{(2)} - \mathrm{i}/2} \\ \prod_{k \neq j}^{M_r} \frac{\lambda_j^{(r)} - \lambda_k^{(r)} - \mathrm{i}}{\lambda_j^{(r)} - \lambda_k^{(r)} + \mathrm{i}} &= \prod_{k=1}^{M_{r-1}} \frac{\lambda_j^{(r)} - \lambda_k^{(r-1)} - \mathrm{i}/2}{\lambda_j^{(r)} - \lambda_k^{(r-1)} + \mathrm{i}/2} \prod_{k=1}^{M_{r+1}} \frac{\lambda_j^{(r)} - \lambda_k^{(r+1)} - \mathrm{i}/2}{\lambda_j^{(r)} - \lambda_k^{(r+1)} + \mathrm{i}/2} \end{aligned} \tag{7}$$

Here $j = 1, \ldots, M_r$ with $r = 2, \ldots, N-1$ and $M_N = 0$. The eigenenergies of H are given by

$$E = -\sum_{j=1}^{M_1} \frac{1}{(\lambda_j^{(1)})^2 + 1/4} \tag{8}$$

The Hamiltonian (6) can be interpreted as that of a spin-S chain by the identification

$$P_{i,\,i+1}^{(N)} = \sum_{\alpha=0}^{N-1} (-)^{N-1-\alpha} \prod_{\beta \neq \alpha}^{N-1} \frac{\mathbf{S}_i^{(N)} \cdot \mathbf{S}_{i+1}^{(N)} - x_\beta}{x_\alpha - x_\beta} \tag{9}$$

where $x_\alpha = \frac{1}{2}\alpha(\alpha+1) - S(S+1)$ and $N = 2S+1$.[7] The components of the spin operator $\mathbf{S}^{(N)}$ are defined by (4). In the simplest case, $S = \frac{1}{2}$, one recovers the Heisenberg model,

$$P_{i,\,i+1}^{(2)} = \tfrac{1}{2}(1 + \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) \tag{10}$$

in terms of the Pauli matrices $\boldsymbol{\sigma}$.

As an historical aside, we note that the $su(3)$ case of the Bethe equations (7) appeared 30 years ago in a paper by Baxter, with regard to the Bethe Ansatz solution of a colouring problem on the honeycomb lattice.[24] The $su(3)$ chain, in terms of spin-1 operators, was first solved by Uimin.[25]

3. LADDERS

A key point in the construction is that for every factor p of N, the matrix $E_{\alpha\beta}$ can be interpreted as acting on $\mathbb{C}^p \otimes \mathbb{C}^{N/p}$, i.e.,

$$E_{\alpha\beta}^{(N)} = E_{\alpha'\beta'}^{(p)} \otimes E_{\alpha''\beta''}^{(N/p)} \tag{11}$$

where $\alpha = (N/p)(\alpha' - 1) + \alpha''$. It follows that the permutator (5) can be rewritten as

$$\begin{aligned} P^{(N)} &= \sum_{\alpha',\, \beta' = 1}^{p} \; \sum_{\alpha'',\, \beta'' = 1}^{N/p} E_{\alpha'\beta'} \otimes E_{\alpha''\beta''} \otimes E_{\beta'\alpha'} \otimes E_{\beta''\alpha''} \\ &= \{ P^{(p)} \otimes P^{(N/p)} \} \end{aligned} \tag{12}$$

Here, the brackets indicate that we should order the factors in the tensor product in definition (5) of $P^{(N)}$ according to the first line in (12). Accordingly, via the correspondence (9), the local Hamiltonian may be interpreted as that of a ladder with spin-$(p-1)/2$ degrees of freedom on one leg and spin-$(N-p)/2p$ on the other leg. In the case of $N=4$, $p=2$ this amounts to

$$H=\sum_{i=1}^{L} \tfrac{1}{4}(1+\boldsymbol{\sigma}_{i,1}\cdot\boldsymbol{\sigma}_{i+1,1})(1+\boldsymbol{\sigma}_{i,2}\cdot\boldsymbol{\sigma}_{i+1,2}) \tag{13}$$

In general, any factorization of N,

$$N=\prod_{j=1}^{q} p_j^{m_j}, \qquad \sum_{j=1}^{q} m_j=n \tag{14}$$

will give rise to an n-leg mixed spin ladder, with spin-$(p_j-1)/2$ on m_j legs, with Hamiltonian

$$H=\sum_{i=1}^{L}\left\{\bigotimes_{j=1}^{q}\bigotimes_{k=1}^{m_j} P_{i,i+1}^{(p_j)}\right\} \tag{15}$$

In fact, there are $(n!/(m_1!\cdots m_q!))$ equivalent ladders depending on the ordering of the different spin degrees of freedom on the legs. Again, in the simple case of $N=2^n$ and $p_1=2$, $m_1=n$, one finds,

$$H=\sum_{i=1}^{L}\frac{1}{2^n}\prod_{l=1}^{n}(1+\boldsymbol{\sigma}_{i,l}\cdot\boldsymbol{\sigma}_{i+1,l}) \tag{16}$$

In the following we will no longer need to specify if some factors p_j are equal and we will therefore drop the detailed notation (14). We will write $N=\prod_{j=1}^{n} p_j$ where p_j's are allowed to be the same.

It is worth mentioning that the above procedure can also be carried out for fermionic ladders that are obtained from a graded permutation operator.[9, 10] In such a way one may construct mixed extended $t-J$ and Hubbard ladder models.

The simplicity of the above construction lies in the simple factorisation (12) property of the permutator. It is possible however to construct anisotropic ladder models from R-matrices related to the q-deformed $su(N)$ algebras.

4. RUNG INTERACTIONS

For any two factors from (15), the product of their respective spin components commutes with (15). Indeed, it can be readily verified using the definitions (1), (4) and (5), that

$$[(S^a)_{i,k}^{(p_k)}\otimes(S^a)_{i,l}^{(p_l)}+(S^a)_{i+1,k}^{(p_k)}\otimes(S^a)_{i+1,l}^{(p_l)},\{P_{i,i+1}^{(p_k)}\otimes P_{i,i+1}^{(p_l)}\}]=0 \tag{17}$$

It thus follows that one can put XYZ type interactions on the rungs which commute with the Hamiltonian (15). This means that the ladder Hamiltonian

$$H=\sum_{i=1}^{L}\left[\left\{\bigotimes_{j=1}^{n} P_{i,i+1}^{(p_j)}\right\}+\sum_{j<k}^{n}\sum_{a=1}^{3} J_a(j,k)(S^a)_i^{(p_j)}\otimes(S^a)_i^{(p_k)}\right] \tag{18}$$

is integrable. A magnetic field term may be added to this Hamiltonian without destroying the integrability.

In general the rung couplings and magnetic field appear as chemical potential terms in the Bethe Ansatz solutions, i.e., they do not appear in the Bethe equations (7), only as additional terms in the eigenvalue expression (8). This is typical of this class of ladder model.

5. EXAMPLES

The result (18) contains previously known examples. For $N=2^n$ and the choice $J_a(j,k)=2J\delta_{k,j+1}$ it reduces to the n-leg spin-$\frac{1}{2}$ model[7]

$$H=\sum_{i=1}^{L}\left[\frac{1}{2^n}\prod_{l=1}^{n}(1+\boldsymbol{\sigma}_{i,l}\cdot\boldsymbol{\sigma}_{i+1,l})+\frac{1}{2}J\sum_{l=1}^{n}\boldsymbol{\sigma}_{i,l}\cdot\boldsymbol{\sigma}_{i,l+1}\right] \tag{19}$$

For $n=2$ this is the model discussed by Wang.[5]

Another interesting example is a mixed spin ladder, with spin-$\frac{1}{2}$ on one leg and spin-1 on the other. The Hamiltonian is given by

$$H=\sum_{i=1}^{L}\left\{\tfrac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_{i+1})[(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^2+\mathbf{S}_i\cdot\mathbf{S}_{i+1}-1]+J\boldsymbol{\sigma}_i\cdot\mathbf{S}_i\right\} \tag{20}$$

where we have taken the rung interactions to be isotropic. This model is based on the $su(6)$ Bethe equations. For this model the two-site rung Hamiltonian consists of a doublet and a quadruplet, so the model remains critical for large rung coupling. On the other hand, the mixed spin-$\frac{1}{2}$ spin-$\frac{3}{2}$ model, with Hamiltonian

$$\begin{aligned}H=\sum_{i=1}^{L}\{\tfrac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_{i+1})[\tfrac{2}{9}(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^3+\tfrac{11}{18}(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^2\\ -\tfrac{9}{8}\mathbf{S}_i\cdot\mathbf{S}_{i+1}-\tfrac{67}{32}]+J\boldsymbol{\sigma}_i\cdot\mathbf{S}_i\}\end{aligned} \tag{21}$$

exhibits a transition to a massive phase at some finite rung coupling J.

The other example we mention here is the spin-1 ladder, with Hamiltonian

$$H = \sum_{i=1}^{L} \{[(\mathbf{S}_{i,1} \cdot \mathbf{S}_{i+1,1})^2 + \mathbf{S}_{i,1} \cdot \mathbf{S}_{i+1,1} - 1][(\mathbf{S}_{i,2} \cdot \mathbf{S}_{i+1,2})^2 + \mathbf{S}_{i,2} \cdot \mathbf{S}_{i+1,2} - 1] + J\mathbf{S}_{i,1} \cdot \mathbf{S}_{i,2}\} \quad (22)$$

This model becomes massive at $J_c = 4$, with the gap opening up linearly with J.

We hope to report on the physical properties of these new models in the near future.

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